Sparse Bayesian Learning: A Beamforming and Toeplitz Approximation Perspective

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Some References

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1. M. Tipping, Sparse Bayesian learning and the relevance vector machine, J. Machine Learning Research, vol. 1, pp. 211-244, 2001.

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7. M. Al-Shoukairi, M. and B. Rao, Sparse Signal Recovery Using MPDR Estimation, 2019 ICASSP.

Outline

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- Connection between SBL and MPDR beamforming
- Uniform Linear Arrays and Toeplitz Matrix Approximation
- Nested Arrays and more sources than sensors

Beamforming and MPDR in Array Processing

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Narrow-band signal model for an array with N sensors.

$$\mathbf{y}(\omega_c, n) = \mathbf{V}(\omega_c, \mathbf{k}_0)x_0[n] + \sum_{l=1}^{D-1} \mathbf{V}(\omega_c, \mathbf{k}_l)x_l[n] + \mathbf{Z}[n], n = 1, .., L$$

ω_c is the carrier frequency, $\mathbf{k}_l$ is the source direction, and $\mathbf{V}(\omega_c, \mathbf{k})$ is the array manifold and is the response of the array to a source at direction $\mathbf{k}$.

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So array manifold can be denoted by $\mathbf{V}(\omega_l)$.

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\begin{aligned}
\mathbf{y}(n) &= \sum_{l=0}^{D-1} \mathbf{V}(\omega_l) x_l[n] + \mathbf{Z}[n] \\
&= [\mathbf{V}_0, \mathbf{V}_1, \ldots, \mathbf{V}_{D-1}] \begin{bmatrix} x_0[n] \\ x_1[n] \\ \vdots \\ x_{D-1}[n] \end{bmatrix} + \mathbf{Z}[n] \\
&= \mathbf{V}\mathbf{x}[n] + \mathbf{Z}[n]
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where $\mathbf{V} \in C^{N \times D}$, and $\mathbf{x}[n] \in C^{D \times 1}$.

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where $\mathbf{V} \in C^{N\times D}$, and $\mathbf{x}[n] \in C^{D\times 1}$.

Assumptions: $E(\mathbf{x}[n]) = \mathbf{0}_{D\times 1}$, $E(\mathbf{x}[n]\mathbf{Z}^H[n]) = \mathbf{0}_{D\times N}$ and temporally uncorrelated.

Covariance Matrices of interest

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Data Covariance Matrix

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$\mathbf{R}_y \in C^{N\times N}$, $\mathbf{R}_s \in C^{N\times N}$, $\mathbf{R}_x \in C^{D\times D}$, and $\mathbf{V} \in C^{N\times D}$

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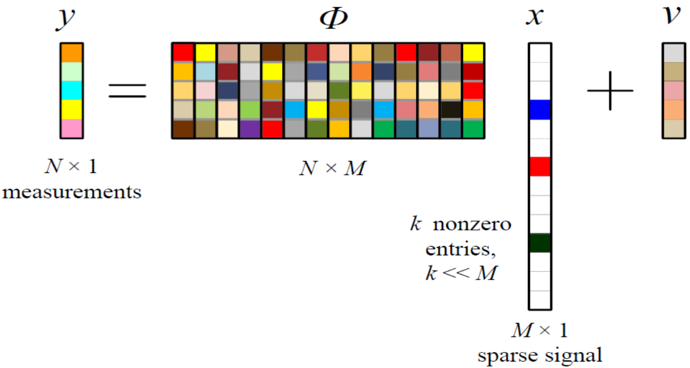
For ULA and uncorrelated sources, $\mathbf{R}_y$ is a Toeplitz matrix.

Sparse Signal Recovery (SSR)



- y is a $N \times 1$ measurement vector and x is $M \times 1$ desired vector which is sparse with k non zero entries.
- Φ is $N \times M$ dictionary matrix where $M >> N$.

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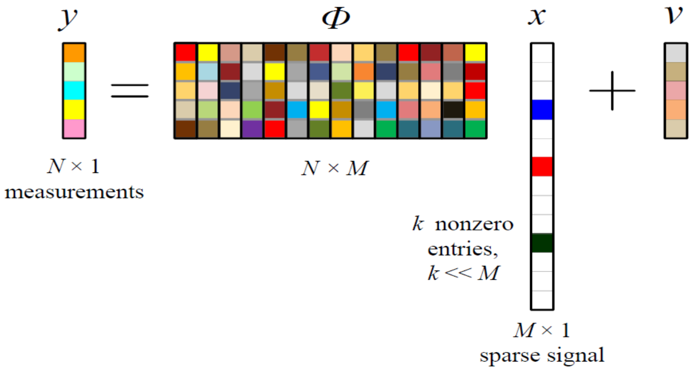
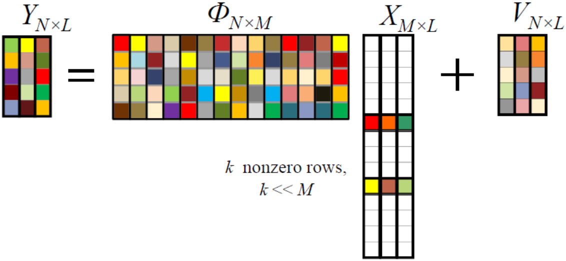


- y is a $N \times 1$ measurement vector and x is $M \times 1$ desired vector which is sparse with k non zero entries.
- Φ is $N \times M$ dictionary matrix where $M >> N$. For array processing, Φ is obtained by employing a grid, i.e. lth column is $\mathbf{V}(\omega_l)$.

Multiple Measurement Vectors (MMV)

- Model



- Multiple measurements: L measurements
- Common Sparsity Profile: k nonzero rows

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Pick a direction of interest ω_s, and select the linear combining weights W such that $r[n] = W^H \mathbf{y}[n]$ contains mostly the signal from direction ω_s

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BF with Null constraints: Incorporate constraints, usually nulls in certain directions, in the BF design.

ULA and Beamforming in Pictures

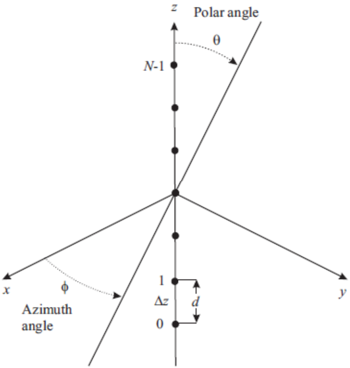

Figure: ULA on the z-axis

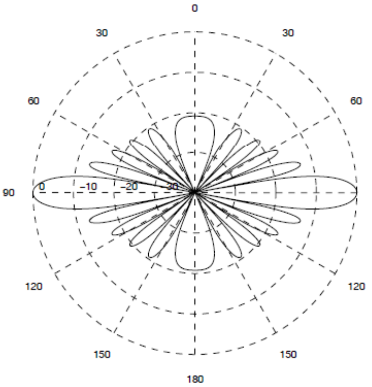

Figure: BF pattern(polar plot, $N = 11$)

Minimum power distortionless response (MPDR) beamformer

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MPDR BF design

$$\min_W W^H \mathbf{R}_y W \text{ subject to } W^H \mathbf{V}_s = 1.$$

Solution: $W_{mpdr} = \frac{1}{\mathbf{V}_s^H \mathbf{R}_y^{-1} \mathbf{V}_s} \mathbf{R}_y^{-1} \mathbf{V}_s$

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Benefit:

- $\mathbf{R}_y$ is easier to determine making it computationally attractive

$$\mathbf{R}_y \approx \frac{1}{L} \sum_{n=1}^{L-1} \mathbf{y}[n]\mathbf{y}^H[n]$$

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Spatial Power Spectrum using MPDR:

$$P_{mpdr}(\omega_s) = E(|W_{mpdr}^H \mathbf{y}[n]|^2) = W_{mpdr}^H \mathbf{R}_y W_{mpdr} = \frac{1}{\mathbf{V}_s^H \mathbf{R}_y^{-1} \mathbf{V}_s}$$

MPDR intuition

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$$\begin{aligned} W^H \mathbf{y}[n] &= W^H \mathbf{V}_s x_s[n] + W^H \mathbf{l}[n] \\ &= \underbrace{x_s[n]}_{\text{distortionless constraint}} + q[n], \quad \text{where} \quad q[n] = W^H \mathbf{l}[n] \end{aligned}$$

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If the interference is correlated with the desired source, we have cancellation and the desired source is not preserved.

MPDR: Uncorrelated sources

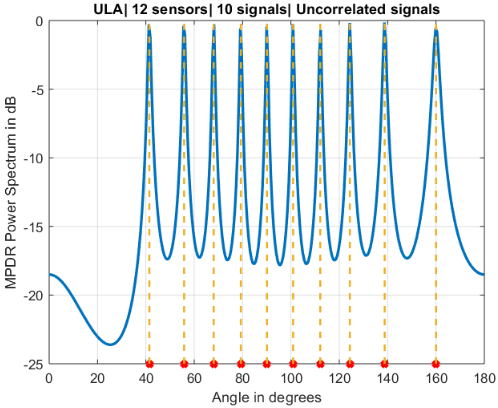
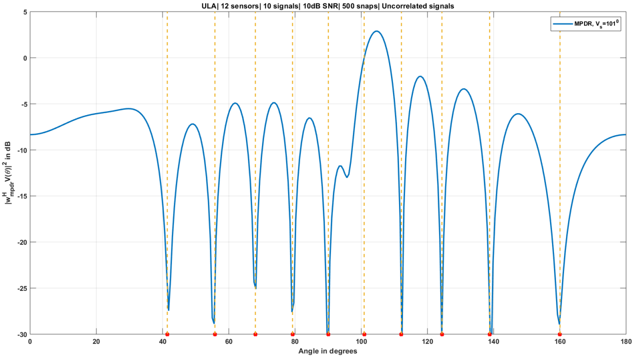


Figure: nsignals = 10 and nsensors = 12

MPDR: Beamformer for uncorrelated sources



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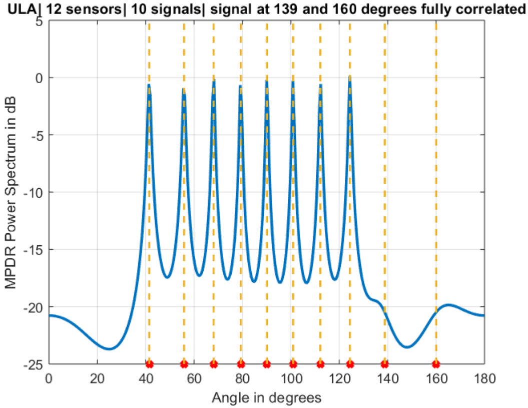

Figure: nsignals = 10, two correlated, and nsensors = 12

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Theorem

A density $p(x)$ which is symmetric with respect to the origin, can be represented by a GSM iff $p(\sqrt{x})$ is completely monotonic on $(0, \infty)$.

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$$\mathbf{y} = \Phi\mathbf{x} + v$$

SBL uses a Bayesian framework with a separable prior $p(\mathbf{x}) = \Pi p(x_i)$.

Gaussian Scale Mixtures (GSM)

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Definition

A function $f(x)$ is completely monotonic on (a, b) if $(-1)^n f^{(n)}(x) \geq 0, \; n = 0, 1, .., $ where $f^{(n)}(x)$ denotes the nth derivative

GSM Hierarchy Viewpoint

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$$\begin{aligned}\mathrm{Kurt}(X) &= E(X^4) - 3(E(X^2))^2 = 3E(\gamma^2) - 3(E(\gamma))^2 \\ &= 3(E(\gamma^2) - (E(\gamma))^2) = 3\mathrm{Var}(\gamma) \geq 0\end{aligned}$$

Examples of Gaussian Scale Mixtures

Laplacian density

$$p(x;\beta) = \frac{1}{2\beta} exp(-\frac{|x|}{\beta})$$

Scale mixing density: $p(\gamma) = \frac{1}{2\beta^2}\exp(-\frac{1}{2\beta^2}\gamma), \gamma \geq 0.$

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$$p(x;a,b) = \frac{b^a\Gamma(a+1/2)}{(2\pi)^{0.5}\Gamma(a)}\frac{1}{(b+x^2/2)^{a+1/2}}$$

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Generalized Gaussian

$$p(x;p) = \frac{1}{2\Gamma(1+\frac{1}{p})}e^{-|x|^p}$$

Scale mixing density: Positive alpha stable density of order $p/2$.

Two Options for Estimation with GSM priors

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MAP Estimation (Type I)

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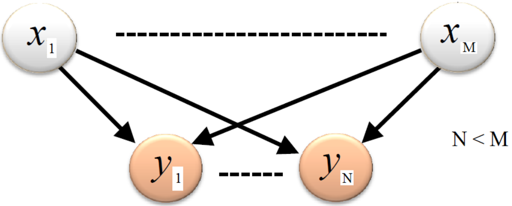
MAP Estimation (Type I)

Evidence Maximization (Type II)

MAP Estimation Framework (Type I)



Problem Statement

$$\hat{x} = \arg\max_{x} p(x|y) = \arg\max_{x} p(y|x)p(x) = \arg\max[\log p(y|x) + \log p(x)]$$

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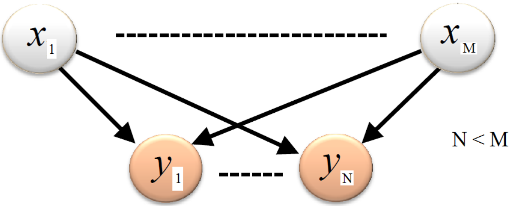

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Can derive a general version that includes many past reweighted algorithms using a generalized-t distribution in one setting[1]

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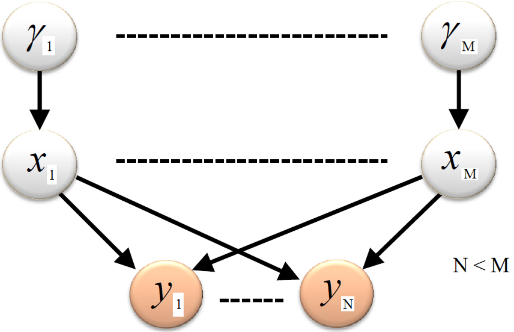

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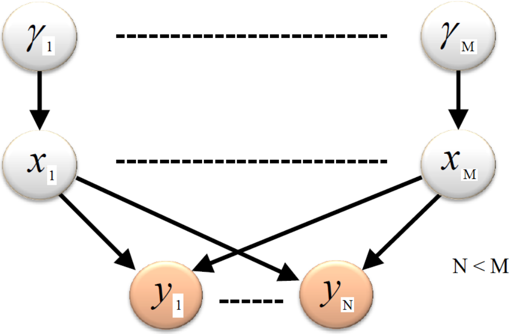

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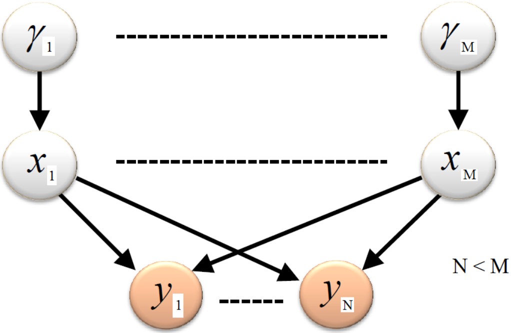

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This leads to Sparse Bayesian Learning (SBL).

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- Maximizing the true posterior mass over the subspaces spanned by non zero indexes instead of looking for the mode.

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Then $p(y|\gamma) = N(y; 0, \Sigma_y)$, where $\Sigma_y = \sigma^2 I + \mathbf{A}\Gamma\mathbf{A}^T$,

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For MMV

$$\hat{\gamma} = \arg\min_{\gamma} \log|\Sigma_y| + \text{Trace } \Sigma_y^{-1}\ \hat{\mathbf{R}}_y - \frac{2}{L}\sum_i \log p(\gamma_i)$$

where $\hat{\mathbf{R}}_y = \frac{1}{L}\sum_{n=1}^{L} \mathbf{y}[n]\mathbf{y}^T[n]$

Algorithmic Variants

- Fixed Point iteration based on setting the derivative of the objective function to zero (Tipping)
- Expectation-Maximization (EM) Algorithm
- Sequential search for the significant γ's (Tipping and Faul)
- Majorization-Minimization based approach (Wipf and Nagarajan)
- Reweighted ℓ_1 and ℓ_2 algorithms (Wipf and Nagarajan)
- Approximate Message Passing (AlShoukairi, Schniter and Rao)

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The conditional mean and variance computation constitute a LMMSE BF!

EM algorithm: Updating γ

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Treating $(\mathbf{y}, \mathbf{x})$ as complete data and vector $\mathbf{x}$ as hidden variable.

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&= \text{argmin}_\gamma \mathbb{E}_{x|y;\gamma^k}[\sum_{i=1}^{M} \left(\frac{x_i^2}{2\gamma_i} + \frac{1}{2}\log \gamma_i \right) - \log p(\gamma)]
\end{aligned}$$

The optimization involves M scalar optimization problems of the form

$$J(\gamma_l) = \frac{E(x_l^2|\mathbf{y}, \gamma^k)}{2\gamma_l} + \frac{1}{2}\log \gamma_l - \log p(\gamma_l)$$

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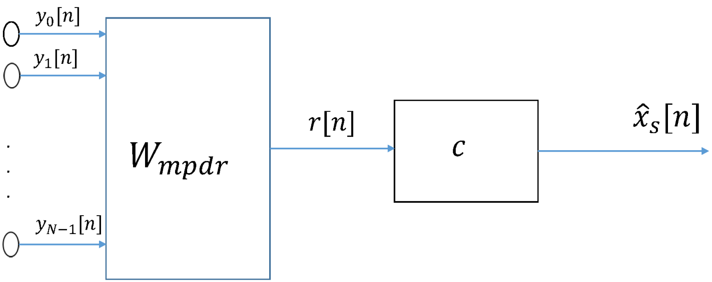
Source powers updated in an iterated manner

Connection between SBL and MPDR



LMMSE Beamformer viewed as a MPDR beamformer followed by scaling for the uncorrelated source case

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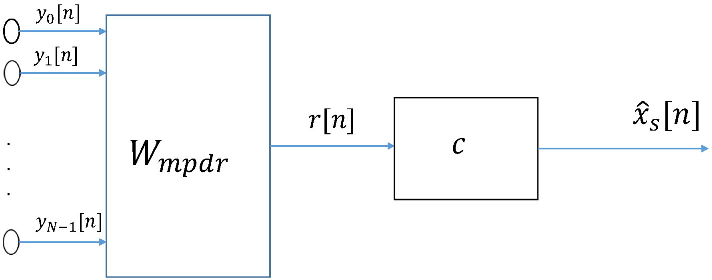

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Note: $r[n] = x_s[n] + q[n]$

MPDR view of SBL

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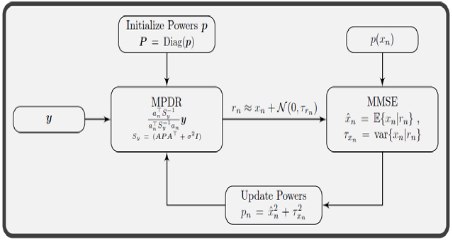
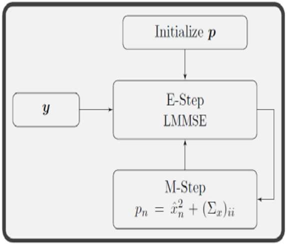


(a) MPDR Based Algorithm



(b) EM-SBL Algorithm

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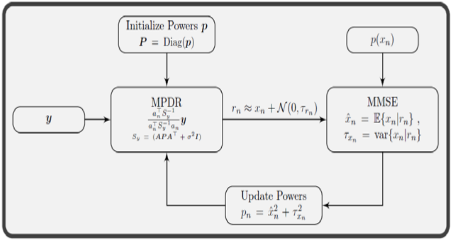

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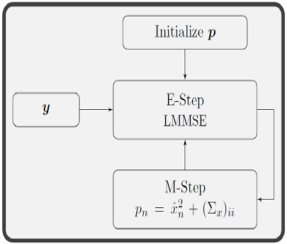

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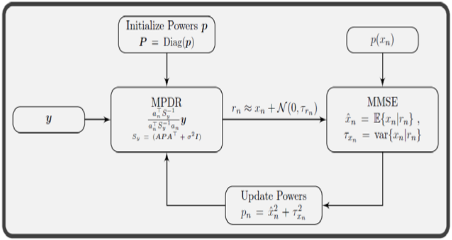

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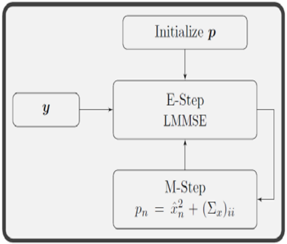

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- When $p(x_n) = \mathcal{N}(0, p_n)$ the MPDR + MMSE steps are equivalent to the LMMSE step in the EM SBL and the two algorithm are equivalent.
- More general priors can be used within the MPDR framework. The EM-SBL has a closed form solution for the GSM prior only.

MPDR versus SBL

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4. Repeat steps 2 and 3 till convergence

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The uncorrelated source model allows SBL to be largely unaffected by correlated sources.

MPDR: Uncorrelated sources

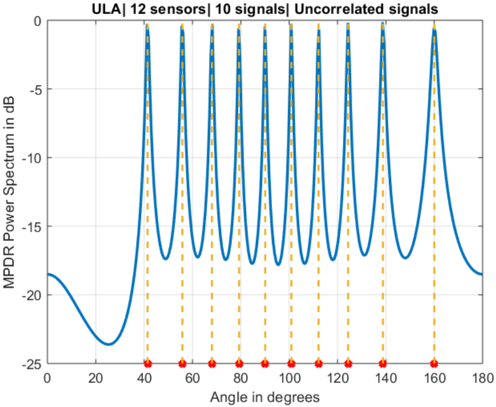

Figure: nsignals = 10 and nsensors = 12

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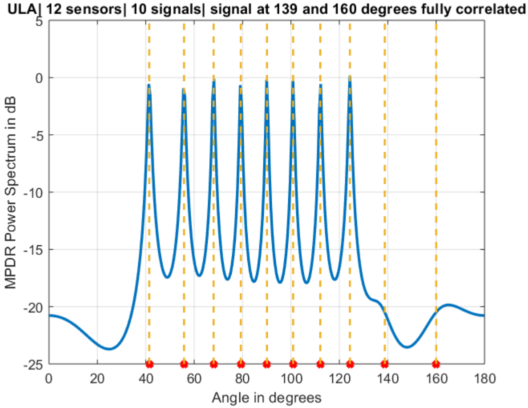


Figure: nsignals = 10, two correlated, and nsensors = 12

Correlated signals with MUSIC

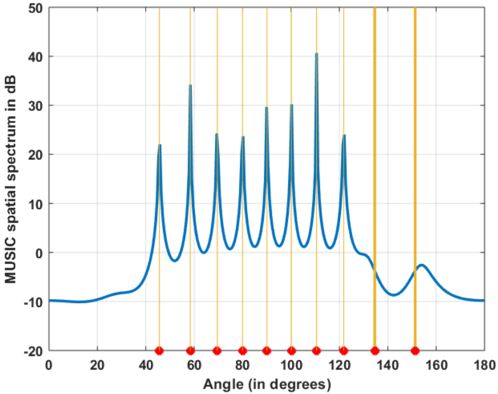
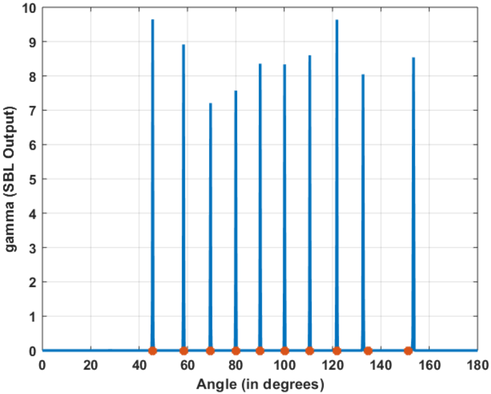


Figure: MUSIC: nsignals = 10 (Last two correlated), nsensors = 12

SBL to the rescue!



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Beam Space processing and Nested Arrays

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- Could be chosen to cover a region in space
- Could be used to thin the array (low complexity sensor array)

Inspiration from Sister Field: Array Signal Processing

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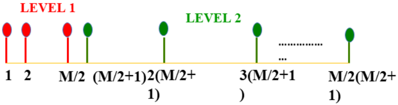
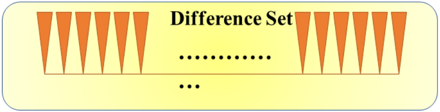
Nested Arrays: Structure & Properties





$M^2/2+M-1$ elements

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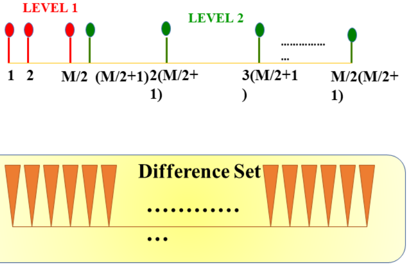

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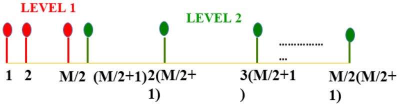

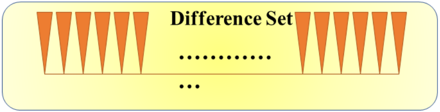

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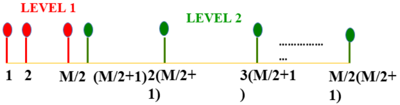

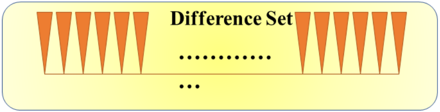

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- Drastically increases the spatial degrees of freedom
- For source localization, this results in estimating location of more sources than sensors!

DoA Estimation with Nested Arrays[2]

With Nested Arrays, we can transform an order-N covariance matrix into order-$\frac{N^2+2N}{4}$ Toeplitz matrix

Transformation:

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Consider as an example $N = 4$, nested array sensor positions: $\{1, 2, 3, 6\}$

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Number of DoAs that can be estimated is $5 > 3$ (for ULA)

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Figure: Nested array with 12 sensors

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Nested Arrays: SBL robustness

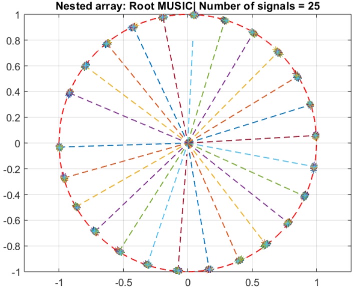

Figure: MUSIC

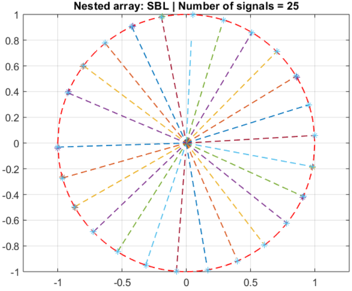

Figure: SBL

Nested Arrays: SBL efficiency

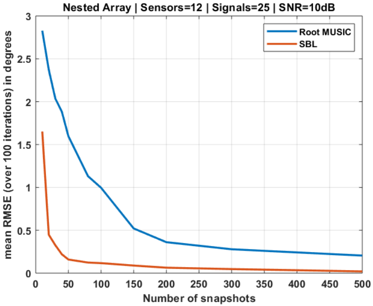

Figure: nsens $= 12$, nsignals $= 25$, SNR $= 10$dB, nsnapshots $= 500$

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  - Enables an approach to deal with more intractable inference problems
- Discussed Uniform Linear Arrays and Toeplitz Matrix Approximation property of SBL
- Shown effectiveness of SBL for Nested Arrays to identify more sources than sensors